UNIT – I

Section: 12

Topological Spaces and Continuous Functions

Definition:

A topology on a set X is a collection τ of subsets of X having the following properties:

i) ϕ and X are in τ (ϕ, X $\in \tau$)
ii) The union of the elements of any sub collection of τ is in τ
iii) The intersection of the elements of any finite sub collection of τ is in τ

A set X for which a topology τ has been specified is called a Topological spaces.

Definition:

If X is any set the collection of all subsets of X is a topology on X which is called the discrete topology.

The collection consisting of ϕ, X only is also a topology on X. We say that it is the indiscrete topology (or) trivial topology.

Definition:

Suppose that τ and τ' are two topologies on a given set X. If $\tau' \supset \tau$, we say that τ' is finer than τ, if τ' properly contains τ, we say that τ' is strictly finer than τ.

We also say that τ is coaser than τ', or strictly coaser is these two respective conditions.

We say τ is comparable with τ' if either $\tau' \supset \tau$ (or) $\tau \supset \tau'$

Section: 13

Basis for a topology

Definition:

If X is a set, a basis for a topology on X is a collection $\mathcal{B}$ of subsets of X called basis elements such that ,

i) for each $x \in X$, there is atleast one basis element $\mathcal{B}$ containing x

ii) if x belongs to the intersection of two basis elements B_1 and B_2 then there is a basis elements $B_3 \in \mathcal{B}$ Such that $x \in B_3 \subset B_1 \cap B_2$

Definition:

If $\mathcal{B}$ satisfies there two conditions then we define a topology τ generated by $\mathcal{B}$ as follows:

. A subset U of X is said to be open in X. (ie) To be an element of τ) if for each x∈U there is a basis element B∈Ɓ such that x∈B and BCU, τ = {UCX ⁄ x∈U=> there exists B∈Ɓ such that x∈BCU}

Now, we prove that the collection τ defined above is a topology.

i) The empty set ϕ∈τ, since it satisfies the definining condition of openness acurously.

Let x∈X

Since Ɓ is a basis, there is a basis element B∈Ɓ such that x∈BCX

ii) Let $\{U_a\}$ be a collection of members of τ

To prove UU_a∈ τ

Let $x \in UU_a$

$x \in U_a$ for some α

since $U_a \in \tau$ and $x \in U_a$ there is some basis element B∈Ɓ such that $x \in BCU_a$

ie) $x \in BCU_aCUU_a$

∴ $UU_a \in \tau$

iii) Let $U_1, U_2, \ldots\ldots\ldots, U_n \in \tau$

To prove : $\bigcap_{i=1}^{n} U_{\mathrm{i}} \in \tau$

To prove : The result by induction on n

When n=1, the result is obviously true

$n=2$, let $x \in U_1 \cap U_2$

$\Rightarrow x \in U_1$ and $x \in U_2$

since $x \in U_1$ then there exist $B_1 \in \mathcal{B}$ such that $x \in B_1 C U_1$

since $x \in U_2$ then there exist $B_2 \in \mathcal{B}$ such that $x \in B_2 C U_2$

$\therefore x \in B_1 \cap B_2 C U_1 \cap U_2$

Since $\mathcal{B}$ is a basis then there exist $B_3 \in \mathcal{B}$ such that $x \in B_3 C B_1 \cap B_2$

$x \in B_1 \cap B_2 C U_1 \cap U_2$

$\therefore U_1 \cap U_2 \in \tau$

Assume that the result is true for n-1

$\therefore U_1 \cap U_2 \cap \ldots\ldots \cap U_{n-1} \in \tau$

Now, $U_1 \cap U_2 \cap \ldots\ldots\ldots \cap U_n = (U_1 \cap U_2 \ldots\ldots\ldots \cap U_{n-1}) \cap U_n$

By induction hypothesis, $U_1 \cap U_2 \cap \ldots\ldots U_{n-1} \in \tau$

By the above part, $(U_1 \cap U_2 \cap \ldots\ldots U_{n-1}) \cap U_n \in \tau$

$\therefore U_1 \cap U_2 \cap \ldots\ldots \cap U_n \in \tau$

Hence $\bigcap_{i=1}^{n} U_i \in \tau$

$\therefore \tau$ is topology on X.

Lemma 13.1

Let X be a set. Let Ɓ be a basis for a topology τ on X. Then τ equals the collection of all unions of elements of Ɓ.

Proof

Given a collection of all elements of Ɓ, they are also elements of τ.

Since τ is a topology, their union is in τ.

Conversely, let $U \epsilon \tau$

Since Ɓ is a basis for τ, for each $X \epsilon U$, then there exists $B_x \epsilon$Ɓ such that $x \epsilon B_x C U$

∴ $U = UB_x$, $B_x \epsilon$Ɓ

∴ U equals to the union of elements of Ɓ

Lemma 13.2

Let X be a topological space. Suppose that Ç is a collection of open sets of X such that for each open set U of X and each x in U, there is an element c of Ç such that $x \epsilon c C U$. Then Ç is a basis for the topology of X.

Proof

Claim Ç is a basis

i) let $x \epsilon X$

since X is open by the definition of Ç, there exists $c \epsilon Ç$ such that $x \epsilon c C X$

ii) let $c_1, c_2 \epsilon Ç$

let $x \epsilon c_1 \cap c_2$

since c_1 and c_2 are open, $c_1 \cap c_2$ is open

By the definition of ς, there exists $c_3 \epsilon ς$ such that $x \epsilon c_3 C c_1 \cap c_2$

∴ Ç is a basis

Let τ be a given topology on X

Let τ' be the topology generated by Ç

Claim $\tau = \tau'$

Let $U \epsilon \tau$

Then by definition of Ç, for each $x \epsilon U$ there exists $c_x C U$

∴ $U = U c_x$

∴ $U \epsilon \tau'$

∴ $\tau C \tau'$-----------❶

Conversely, let $W \epsilon \tau'$

∴ $W = U c_a$, $c_a \epsilon Ç$

Since $c_a \epsilon Ç$, we have $c_a \epsilon \tau$ for all α

Since τ is a topology, $U c_a \epsilon \tau$

∴ $W \epsilon \tau$

$\therefore$ $\tau' \subset \tau$-----------❷

Hence $\tau=\tau'$ (by ❶ &❷)

Lemma 13.3

Let $\mathcal{B}$ and $\mathcal{B}'$ be basis for the topologies τ and τ' respectively on X. Then the following are equivalent

i) τ' is finer than τ

ii) for each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$

Proof

(ii)=> (i)

To prove : τ' is finer than τ

Ie) To prove $\tau' \supset \tau$

Let $U \in \tau$

Let $x \in U$

Since $\mathcal{B}$ is a basis, there exist $B \in \mathcal{B}$ such that $x \in B \subset U$

Since by condition (ii), there exists $B' \in \mathcal{B}'$ such that $x \in B' \subset B \subset U$

$\therefore$ $U \in \tau'$

$\therefore$ τ' is finer than τ.

(i)=> (ii)

For each $x \in X$, there is a basis element $B \in \mathcal{B}$ containing x

Since $\mathcal{B}$ is a basis for a topology τ, $B \in \mathcal{B} \Rightarrow B \in \tau$

Since $\tau' \supset \tau$, $B \in \tau'$ and $x \in B$

$\therefore$ There exists $B' \in \mathcal{B}'$ such that $x \in B' \subset B$

Hence (i) and (ii) are equivalent

Definition:

If $\mathcal{B}$ is the collection of all open intervals in the real line

Ie) $\mathcal{B} = \{ (a,b) / a, b \in R \}$

Then the topology generated by $\mathcal{B}$ is called the standard topology on the real line R. Unless specified, R is given the standard topology.

Definition:

If $\mathcal{B}'$ is the collection of all half open intervals of the form, $\mathcal{B}' = \{[a, b) / a \leq x < b\}$, then the topology generated by $\mathcal{B}'$ is called the lower limits topology in R. When R is given as lower limit topology. We denote it by R_l.

Definition:

Let k denote the set of all numbers of the form $1/n$, $n \in Z_+$ and $\mathcal{B}'$ be the collection of all open intervals (a, b). Along with all sets of the form (a, b)-k. The topology generated by Ɓ" is call the k-topology on R. When R is given the k-topology is denoted by R_k.

Lemma 13.4

The topologies of R_l and R_k are strictly finer than the standard topology on R, but are not comparable with one another

Proof

Let τ, τ' and τ" be the topologies of R, R_l and R_k respectively.

Claim τ' is strictly finer than τ

ie) To prove τ'Ɔ τ

Let (a, b) be a basis element for τ and let x∈(a, b)

Then x∈[x, b)C (a, b)

Since [x, b) is a basis element for τ', then by previous lemma

(ii) τ' is finer than τ

Now , [x, b) is a basis element for τ' contains x

Clearly no basis element of the form (a, b) containing x can be contained in [x, b)

∴ τ' is strictly finer than τ

Claim: τ" is strictly finer than τ

Let (a, b) be a basis element of τ containing x

Clearly (a, b) itself a basis element for τ” contains x and contained in (a, b)

∴ τ” is finer than τ

On the other hand given the basis element B= (-1, 1)-K for τ” and the point zero of B, there is no open interval that contains 0 and lies in B

∴ τ” is strictly finer than τ.

Definition:

A sub basis Ș for a topology on X is a collection of subsets of X whose union equals X. The topology generated by the sub basis Ș is defined to be the collection of all unions of finite intersection of elements of Ș

ie) τ= U { $\bigcap_{i=1}^{n} s_i$/ s_i∈ Ș}

Section 14

The order Topology

Definition:

A relation ‘C’ is on a set A is called an order relation (or) simple order (or) linear order if it has the following properties;

1. For every x and y in A, for which $x \neq y$ either x C y or y C x (comparability)
2. For no x in A , the relation x C x hold (non-reflectivity)
3. If x C y or y C z then x C z (transitivity)

Definition:

Let X be a set having a simple relation < (lessthan) given an element a and b of X suchthat $a<b$, there are four subsets of X that are called the intervals determined by a and b. They are following

1. $(a,b) = \{ x / a < x < b\}$
2. $[a,b) = \{ x / a \leq x < b\}$
3. $(a,b] = \{ x / a < x \leq b\}$
4. $[a,b] = \{ x / a \leq x \leq b\}$

Condition (i) is open, condition (iv) is called closed, condition (ii) & (iii) are called half open intervals.

Definition:

Let X be a set with a simple order relation. Assume X has more than one element.

Let Ɓ be the collection of all sets of the following types.

1. All open intervals (a,b) in X.
2. All intervals of the form $[a_0,b)$, where a_0 the smallest element (if any) of X.
3. All intervals of the form $(a,b_0]$, where b_0 is the largest element (if any) of X.

The collection Ɓ is a basis for a topology on X, which is called the ordered topology.

Definition:

If X is an ordered set and a is an element of X, there are four subsets of X that are called the rays determined by a followings

1. $(a, +\infty) = \{ x / x > a \}$
2. $(-\infty, a) = \{ x / x < a \}$
3. $[a, +\infty) = \{ x / x \geq a \}$
4. $(-\infty, a] = \{x / x \leq a \}$

The sets of the first two types are called open rays and the sets of the last two types are called closed rays.

Section 16

The subspace Topology

Definition:

Let X be a topological space with topology τ. If Y is a subset of X, the collection

$\tau_Y = \{ Y \cap U / U \in \tau \}$ is a topology on Y, called the subspace topology.

With this topology Y is called subspace of X, its open sets consist of all intersection of open sets of X with Y.

Lemma 16.1:

If $\mathcal{B}$ is a basis, for the topology of X, then the collection $\mathcal{B}_Y = \{ B \cap Y / B \in \mathcal{B} \}$ is a basis for the subspace topology on Y.

Let $U \cap Y$ be an open set in Y, where U is open.

Let $y \in U \cap Y$.

Then $y \in U$.

Since $\mathcal{B}$ is a basis for X, there exists a basis element $B \in \mathcal{B}$ suchthat $y \in B \subset U$.

$Y \in B \cap Y \subset U \cap Y$.

Since $B \cap Y \in \mathcal{B}_Y$.

$\mathcal{B}_Y$ is a basis for a subspace topology on Y.

Lemma 16.2:

Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.

Since U is open in Y, $U = V \cap Y$ where V is open in X.

Also Y is open in X.

Since V and Y are open in X, then $V \cap Y$ is open in X.

Hence U is open in X.

Theorem 16.3:

If A is a subspace of X and B is a subspace of Y, then the product topology on AxB is the same as the topology of AXB inherits as a subspace of XxY.

The general basis element for the product topology on XxY is UxV , where U is open in X and V is open in Y.

$\therefore$ $(U \times V) \cap (A \times B)$ is the general basis element for the subspace topology on AxB.

Now, $(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$

Since $U \cap A$ and $V \cap B$ are general open sets in A and B respectively.

Then $(U \cap A) \times (V \cap B)$ is a general basis element for the product topology on AxB.

Thus the basis for the subspace topology on AxB and the product topology on AxB are same.

Hence the two topologies are same.

Definition:

Given an ordered set X. A subset Y of X is said to be convex in X if for each pair points $a<b$ of Y, the entire interval (a,b) of points of X lies in y.

Theorem 16.4:

Let X be an order set in the ordered topology. Let y be a subset of X, that is convex in X. Then the order topology on Y is the same as the topology Y inherits as a subspace of X.

Consider the ray $(a,+\infty)$ in X and Y is convex in X.

If $a \in Y$, then $(a,+\infty) \cap Y = \{ x / x \in Y \text{ and } x > a \}$ which is an open ray in the ordered set y.

If $a \notin Y$, then a is either a lower bound on Y or an upper bound on Y.

If a is a lower bound on y, $(a,+\infty) \cap Y$ equals all of Y.

If a is an upper bound on y, then $(a,+\infty) \cap Y$ is equals to ϕ.

Similarly, $(-\infty,a)$ is either an open ray of Y or Y or ϕ.

Since the set $(a,+\infty)\cap Y$ and $(-\infty,a)\cap Y$ form a subbasis for this subspace topology on Y.

Since each is open in the ordered topology and ordered topology contains this subspace topology.

To prove the reverse part

Any open ray of Y equals the intersection of open ray of X

Since the open ray of Y form the subbasis is for the ordered topology on Y.

The ordered topology on Y equals the subbasis is for the ordered topology on Y equals the subspace topology of y as a subspace of X.

Hence the theorem.

Definition:

A map $f:X\to Y$ is said to be and open map if for every open set U of X, the set f(U) is open in Y. ie) If f is an open map, image every open set U in X under f is open in Y.

Section: 17

Closed sets and limit points

Definition:

A subset A of a topological space X is said to be closed if the X-A is open.

Theorem: 17.1

Let X be a topological space. Then the following conditions hold:

1. Ǿ and X are closed.
2. Arbitrary intersection of closed sets are closed.
3. Finite unions of closed sets are closed.

Solution:

1. Since complement of X and Ǿ are open.

   We have X and Ǿ are closed.

2. Let $\{A_\alpha\}_{\alpha £ J}$ be the collection of closed sets.

   By Demorgan's law,

   $X- \cap_\alpha A_\alpha = \cup_\alpha (X - A_\alpha)$

   Since $X - A_\alpha$, for all α£J is open and arbitrary union of open set is open.

   $\therefore \cup_\alpha (X - A_\alpha)$ is open.

   $\therefore \cap_\alpha A_\alpha$ is closed.

3. Let $A_1, A_2, \ldots\ldots\ldots A_n$ be a closed set in X.

Claim: $\bigcup_{i=1}^{n} A_i$ is closed.

Now, X-$\bigcup_{i=1}^{n} A_i$ = $\bigcap_{i=1}^{n}(X - Ai)$ (By Demorgan's law)

Since each X-A_i is open in X. $\bigcap_{i=1}^{n}(X - Ai)$ is open.

That is X-$\bigcup_{i=1}^{n} A_i$ is open.

$\therefore \bigcup_{i=1}^{n} A_l$ is closed

Theorem: 17.2

Let Y be a subspace of X. Then a set A is closed in Y iff it equals the intersection of a closed set of X with Y.

Solution:

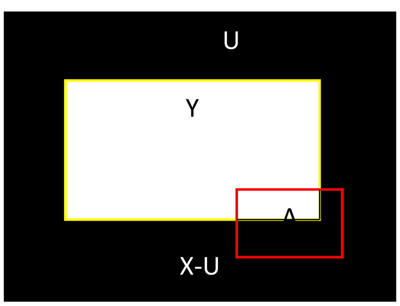
Let A be a closed set in Y.

Then Y-A is open in Y. ($\because$ Y is a subspace)

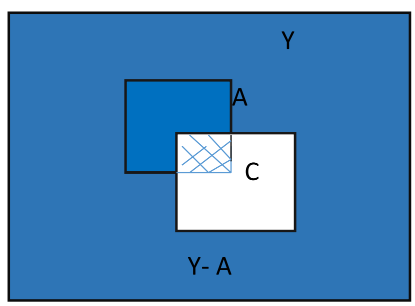
$\therefore$ Y-A = U∩Y, Where U is open in X.

$\therefore$ X-U is closed in X and (X-U)∩Y = A.

(i)



(ii)



$\therefore$ A equals the intersection of a closed set in X with Y.

Conversely,

Let A = C∩Y, where C is closed in X.

To prove: A is closed in Y.

Since C is closed in X, X-C is open in X.

∴ $(X-C) \cap Y$ is open in Y.

$(X-C) \cap Y = Y-A$ which is open in X.

∴ A is closed set in y.

Theorem: 17.3

Let Y be a subspace of X. If A is closed in Y and Y is closed in X then A is closed in X.

Solution:

Let A be a closed set in Y.

$A = C \cap Y$, where C is closed in X.

Since C and Y are closed in X, A is closed in X.

Closed and interior of a set

Definition:

Given a subset A of a topological space X, the interior of A is defined as the union of all open set contained in A.

Definition:

The closure of A is the intersection of all closed sets containing A.

Theorem: 17.4

Let Y be a subspace of X. Let $\bar{A}$ be a subset of Y. Let $\bar{A}$ denote the closure of A in X. Then the closure of A in Y equals $\bar{A} \cap Y$.

Solution:

Let B be the closure of A in Y.

To prove: $B = \bar{A} \cap Y$

Clearly, $\bar{A} \cap Y$ is closed in y. (By theorem 17.2)

Since B is the smallest closed set in Y containing A,

We have $B \subset \bar{A} \cap Y \longrightarrow (1)$

On the other hand, we know that B is closed in Y.

$\therefore B = C \cap Y$, where C is closed in X.

Since B is the closure of A in Y, then C is a closed set

Containing A in X.

Since $\bar{A}$ is the smallest closed set containing A, we have

$\bar{A} \subset C$.

$\therefore \bar{A} \cap Y \subset C \cap Y$

$\bar{A} \cap Y \subset B \longrightarrow (2)$

From (1) and (2), $B = \bar{A} \cap Y$.

Theorem: 17.5

Let A be a subset of the topological space X.

Then (i) $x \in \bar{A}$ iff every open set U containing x intersects A.

(ii) Supposing the topology of X is given by a basis, then $x \in \bar{A}$ iff every basis element B containing x intersects A.

Solution:

(i) We shall prove that $x \notin \bar{A}$ iff there exists an open set U containing x that does not intersects A.

Let $x \notin \bar{A}$.

Then the set $U = X - \bar{A}$ is an open set containing x that does not intersect A.

Conversely,

Let U be an open set containing x that does not intersect A

$\therefore$ X-U is a closed set containing A.

But $\bar{A}$ is the smallest closed set containing A.

$\therefore \bar{A} \subset X-U$

Since $x \notin X-U$, $x \notin \bar{A}$.

(ii) Let $x \in \bar{A}$

To prove: Every basis element B containing x intersects A.

We know that, Every basis element is open.

$\therefore$ Every basis element B containing x also intersects A.

Conversely,

Every basis element B containing x intersects A, so does every open set U containing x, because U contains a basis element that contains x.

By (i), $x \in \bar{A}$.

Theorem: 17.6

Let A be a subset of the topological space X. Let A' be the set of all limit points of A. Then $\bar{A} = A \cup A'$.

Solution:

Let $x \in A \cup A'$

$\Rightarrow x \in A$ (or) $x \in A'$

If $x \in A$, then $x \in \bar{A}$ $\quad (\because A \subset \bar{A})$

If $x \in A'$, then every neighbourhood of x intersects A-{x}.

Then every neighbourhood of x intersects A.

$\therefore x \in \bar{A}$ $\quad (\because$ By theorem 17.5)

$\therefore x \in A \cup A'$

$A \cup A' \subset \bar{A} \longrightarrow (1)$

If $x \in A$ then trivially, $x \in A \cup A'$

If $x \notin A$, then every neighbourhood of x intersects $A-\{x\}$.

($\because x \in \bar{A}$).

Then x is a limit point of A.

$\therefore\ x \in A'$

$\therefore\ x \in A \cup A'$

$\therefore\ \bar{A} \subset A \cup A' \longrightarrow (2)$

From (1) and (2), $\bar{A} = A \cup A'$.

Corollary:

A subset of a topological space is closed iff it contains all its limit points.

Solution:

Let A be a closed subset of a topological space.

We know that $\bar{A} = A \cup A'$ ($\because$ A is closed $\Rightarrow A=\bar{A}$)

$\therefore A' \subset A$

$\therefore$ A contains all its limit points.

Conversely,

Suppose $A' \subset A$

$A \cup A' \subset A \cup A = A$

$A \cup A' \subset A$

$\bar{A} \subset A$, also we know that $A \subset \bar{A}$.

$\therefore A = \bar{A}$

$\therefore$ A is closed.

Hausdorff Space

Definition:

A topological space X is called a hausdorff space if for each pair (x_1, x_2) of distinct points of X, there exists neighbourhoods U_1 and U_2 of x_1 and x_2 respectively that are disjoint.

Theorem: 17.8

Every finite point set in a hausdorff space X is closed.

Solution:

It is enough to prove that every one point set $\{x_0\}$ is closed.

That is, to prove $\{x_0\} = \overline{\{x_0\}}$

Let x be a point of X such that $x \neq x_0$

Since X is a hausdorff space, there exists disjoint neighbourhoods U and V of x and x_0 respectively.

U is a neighbourhood of x that does not intersect $\{x_0\}$

$\therefore\ x \notin \overline{\{x_0\}}$ ($\because$ Theorem: 17.5)

Hence the closure of the set $\{x_0\}$ is $\{x_0\}$ itself.

That is, $\{x_0\} = \overline{\{x_0\}}$

$\therefore$ $\{x_0\}$ is closed.

Theorem: 17.9

Let X be a space satisfying the T_1- axiom. Let A be a subset of X. Then the Point x is a limit point of A iff every neighbourhood of x contains infinitely many points of A.

Solution:

Let x be a limit point of A.

To prove: Every neighbourhood of x contains

Infinitely many points of A.

Suppose, there exists a neighbourhood U of x intersects A in only finitely many points.

We have $U \cap A$ = finite set.

$U \cap A - \{x\} = \{x_1, x_2, \ldots\ldots., x_n\}$ (say)

Since X is a T_1-space, $\{x_1, x_2, \ldots\ldots. x_n\}$ is closed in X.

$\therefore$ $X - \{x_1, x_2, \ldots\ldots.. x_n\}$ is open in X.

$\therefore$ $U \cap \{X - \{x_1, x_2, \ldots\ldots x_n\}\}$ is also a neighbourhood of x which does not intersects $A - \{x\}$.

That is, $[U \cap \{X - \{x_1, x_2, \ldots\ldots x_n\}\}] \cap A - \{x\} = \phi$

∴ x is not a limit point of A, which is a contradiction to our assumption.

Hence every neighbourhood of x contains infinitely many points of A.

Conversely,

Suppose every neighbourhood of x contains infinitely many points of A.

∴ Every neighbourhood of x intersects A-{x}.

∴ x is a limit point of A.

Theorem: 17.10

If X is a hausdorff space, then a sequence of points of X converges to almost one point x.

Solution:

Given that X is a hausdorff space.

Let $\{x_n\}$ be a sequence of points in X.

Let x, y be a two points of X such that $x \neq y$.

Let $\{x_n\}$ converges to x.

To prove: $\{x_n\}$ does not converges to x.

Since $x \neq y$, and X is a hausdorff space there exists disjoint neighbourhoods U and V of x and y respectively.

Since x is a limit point of U and U is a neighbourhood of x, U contains $\{x_n\}$ for all but finitely many values of n.

Hence the neighbourhood V of y cannot contains infinitely many points of $\{x_n\}$.

$\therefore$ $\{x_n\}$ does not converges to y.

Theorem: 17.11

Every simply ordered set is a hausdorff space in the Order topology. The product of two hausdorff spaces is A hausdorff space. A subspace of a hausdorff is a Hausdorff space.

Solution:

Let X, Y be two hausdorff spaces.

Claim: X x Y is a hausdorff space.

Let $x_1 \times y_1$ and $x_2 \times y_2 \in X \times Y$ such that $x_1 \times y_1 \neq x_2 \times y_2$

Case (i)

Let $x_1 \neq x_2$ and $y_1 \neq y_2$.

Then there exists neighbourhoods U_1 and U_2, V_1 and V_2 of (x_1, x_2) and (y_1, y_2) respectively such that $U_1 \cap U_2 = \emptyset$ and $V_1 \cap V_2 = \emptyset$.

Then $(U_1 \times V_1)$ is a neighbourhood of $x_1 \times y_1$ and $(U_2 \times V_2)$ is a

Neighbourhood of x_2 x y_2.

$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) = \emptyset$.

Case (ii)

Let $x_1 = x_2 = x$, and $y_1 \neq y_2$.

Then there exists a neighbourhood U and V of y_1 and y_2

respectively such that $U \cap V = \emptyset$.

Then X x U is a neighbourhood of x_1 x y_1 and X x V is a

Neighbourhood of x_2 x y_2.

$(X \times U) \cap (X \times V) = (X \cap X) \times (U \cap V)$

$= X \times \emptyset$

$= \emptyset$.

Case (iii)

Let $x_1 \neq x_2$, $y_1 = y_2 = y$

Since $x_1 \neq x_2$ there exists a neighbourhood U and V of x_1 and

x_2 respectively such that $U \cap V = \emptyset$.

Then U x Y is a neighbourhood of x_1 x y_1 and V x Y is a

neighbourhood of x_2 x y_2.

Now, $(U \times Y) \cap (V \times Y) = (U \cap V) \times (Y \cap Y)$

$= \emptyset \times Y$

$= \emptyset$

$\therefore$ The product of two hausdorff space is again a hausdorff

Space.

Let Y be a subspace of a hausdorff space.

Let $y_1, y_2 \in Y$ and $y_1 \neq y_2$

Since Y is a subspace of X, $y_1, y_2 \in X$ and $y_1 \neq y_2$.

Since X is a husdorff space, there exists a neighbourhoods

U and V of y_1 and y_2 in X respectively such that $U \cap V = \emptyset$.

Then $U \cap Y$ is a open set of y_1 in Y and $V \cap Y$ is an open set of

Y_2 in Y.

Now,

$$(U\cap Y) \cap (V\cap Y) = (U\cap V) \cap (Y\cap Y)$$

$$= \emptyset \cap Y$$

$$= \emptyset$$

Hence subspace of a hausdorff space is a hausdorff space.